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# ON THE RELATION BETWEEN THE WEAK PALAIS-SMALE CONDITION AND COERCIVITY BY ZHONG

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ABSTRACT. In this paper, we discuss Zhong's result of that the weak Palais-Smale condition implies coercivity under some assumption in [Nonlinear Anal., 29 (1997), 1421–1431]. We also give a simple proof of Zhong's result. Further we generalize the result in Caklovic, Li and Willem [Differential Integral Equations, 3 (1990), 799–800].

## 1. INTRODUCTION

Throughout this paper we denote by  $\mathbb{N}$  the set of all positive integers and by  $\mathbb{R}$  the set of all real numbers.

Let  $f$  be a function from a Banach space  $X$  into  $(-\infty, +\infty]$ . We recall that  $f$  is called *Gâteaux differentiable* at  $x \in X$  with  $f(x) \in \mathbb{R}$  if there exists a continuous linear functional  $f'(x)$  such that

$$\lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t} = \langle f'(x), y \rangle$$

holds for every  $y \in X$ .  $f$  is said to be *coercive* if

$$\lim_{r \rightarrow \infty} \inf_{\|x\| \geq r} f(x) = \infty$$

holds. Also,  $f$  is said to satisfy the *weak Palais-Smale condition* [17] if there exists a nondecreasing function  $h$  from  $[0, \infty)$  into itself satisfying  $\int_0^\infty (1/(1+h(t)))dt = \infty$ , and the following condition: Every sequence  $\{x_n\}$  in  $X$  such that  $\{f(x_n)\}$  is bounded and

$$\lim_{n \rightarrow \infty} \|f'(x_n)\| (1 + h(\|x_n\|)) = 0$$

contains a convergent subsequence. This definition seems to be weaker than the definition in [17]. However they are equivalent; see Section 5. In the case of  $h(t) = 0$  for all  $t \in [0, \infty)$ , we call that  $f$  satisfies the *Palais-Smale condition*. In the case of  $h(t) = t$  for all  $t \in [0, \infty)$ , we call that  $f$  satisfies the *Cerami-Palais-Smale condition* [4].

It is well known that the Palais-Smale condition implies coercivity under some assumption; see Brézis and Nirenberg [2], Caklovic, Li and Willem [3] and others. In 1997, Zhong [17] generalized these results and proved that the weak Palais-Smale condition implies coercivity. However the proof is slightly complicated.

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In this paper, we discuss Zhong's result and we also give a simple proof of it. Further we generalize the result in Caklovic, Li and Willem [3]. We also discuss the conditions of the continuity of  $h$ ,  $\int_0^\infty (1/(1+h(t)))dt = \infty$ , and the completeness of  $X$ .

## 2. $\tau$ -DISTANCE

In our discussion, the notion of  $\tau$ -distance plays an important role.

Let  $(X, d)$  be a metric space. Then a function  $p$  from  $X \times X$  into  $[0, \infty)$  is called a  $\tau$ -distance on  $X$  [10] if there exists a function  $\eta$  from  $X \times [0, \infty)$  into  $[0, \infty)$  and the following are satisfied:

- ( $\tau 1$ )  $p(x, z) \leq p(x, y) + p(y, z)$  for all  $x, y, z \in X$ ;
- ( $\tau 2$ )  $\eta(x, 0) = 0$  and  $\eta(x, t) \geq t$  for all  $x \in X$  and  $t \in [0, \infty)$ , and  $\eta$  is concave and continuous in its second variable;
- ( $\tau 3$ )  $\lim_n x_n = x$  and  $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$  imply  $p(w, x) \leq \liminf_n p(w, x_n)$  for all  $w \in X$ ;
- ( $\tau 4$ )  $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$  and  $\lim_n \eta(x_n, t_n) = 0$  imply  $\lim_n \eta(y_n, t_n) = 0$ ;
- ( $\tau 5$ )  $\lim_n \eta(z_n, p(z_n, x_n)) = 0$  and  $\lim_n \eta(z_n, p(z_n, y_n)) = 0$  imply  $\lim_n d(x_n, y_n) = 0$ .

We note that  $\eta$  is strictly increasing in its second variable. We also note that the metric  $d$  is a  $\tau$ -distance on  $X$ . Many useful propositions and examples are stated in [7–16].

Though the following is a corollary of Proposition 2 in [12], we give a proof.

**Proposition 1.** *Let  $(X, d)$  be a metric space with a  $\tau$ -distance  $p$ . Let  $q$  be a function from  $X \times X$  into  $[0, \infty)$ . Suppose that*

- (i)  $q$  satisfies  $(\tau 1)_q$ , i.e.,  $q(x, z) \leq q(x, y) + q(y, z)$  for all  $x, y, z \in X$ ;
- (ii)  $q$  is lower semicontinuous in its second variable;
- (iii)  $q(x, y) \geq p(x, y)$  for all  $x, y \in X$ .

*Then  $q$  is also a  $\tau$ -distance on  $X$ .*

*Proof.* Let  $\eta$  be a function satisfying  $(\tau 2)$ – $(\tau 5)$ . From the assumption (ii),  $(\tau 3)_q$  clearly holds. We assume that  $\lim_n \sup\{q(x_n, y_m) : m \geq n\} = 0$  and  $\lim_n \eta(x_n, t_n) = 0$ . Then from the assumption (iii), we have  $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ . So by  $(\tau 4)$ , we obtain  $\lim_n \eta(y_n, t_n) = 0$ . This is  $(\tau 4)_q$ . Let us prove  $(\tau 5)_q$ . We assume that  $\lim_n \eta(z_n, q(z_n, x_n)) = 0$  and  $\lim_n \eta(z_n, q(z_n, y_n)) = 0$ . Then from the assumption (iii) again, we have  $\lim_n \eta(z_n, p(z_n, x_n)) = 0$  and  $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ . So by  $(\tau 5)$ , we obtain  $\lim_n d(x_n, y_n) = 0$ . This completes the proof.  $\square$

Now, we give the following example.

**Example 1.** Let  $(X, d)$  be a metric space, and  $h$  a nondecreasing function from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1+h(t)))dt = \infty$ . Fix  $z_0 \in X$ . Then functions  $p$  and  $q$  from  $X \times X$  into  $[0, \infty)$  defined by

$$p(x, y) = \int_{d(z_0, x)}^{d(z_0, x) + d(x, y)} \frac{dt}{1 + h(t)} \quad \text{and} \quad q(x, y) = p(x, y) + p(y, x)$$

for all  $x, y \in X$  are  $\tau$ -distances on  $X$ .

*Proof.* We know that  $p$  is a  $\tau$ -distance on  $X$ ; see Proposition 4 in [10]. So, since  $p$  satisfies  $(\tau 1)$ , we have

$$\begin{aligned} q(x, z) &= p(x, z) + p(z, x) \\ &\leq p(x, y) + p(y, z) + p(z, y) + p(y, x) \\ &= q(x, y) + q(y, z) \end{aligned}$$

for  $x, y, z \in X$ . This is  $(\tau 1)_q$ . It is obvious that  $q$  is continuous and  $q(x, y) \geq p(x, y)$  for all  $x, y \in X$ . So by Proposition 1, we have  $q$  is a  $\tau$ -distance on  $X$ .  $\square$

In [10], using the above  $p$ , the author gave the slight generalization and another proof of Zhong's variational principle [17, 18]. In this paper, we use the above  $q$ .

The following is Theorem 4 in [10], which is the  $\tau$ -distance version of Ekeland's variational principle [5, 6]. Of course, this is one of the generalizations of the Banach contraction principle [1].

**Theorem 1.** *Let  $X$  be a complete metric space with a  $\tau$ -distance  $p$ . Let  $f$  be a function from  $X$  into  $(-\infty, +\infty]$  which is proper lower semicontinuous and bounded from below. Then for  $\varepsilon > 0$  and  $u \in X$  with  $p(u, u) = 0$ , there exists  $v \in X$  such that  $f(v) \leq f(u) - \varepsilon p(u, v)$  and  $f(w) > f(v) - \varepsilon p(v, w)$  for all  $w \in X$  with  $w \neq v$ .*

From Example 1 and Theorem 1, we obtain the following.

**Theorem 2.** *Let  $X, d, h, z_0$  be as in Example 1. Suppose that  $X$  is complete. Let  $f$  be a function from  $X$  into  $(-\infty, +\infty]$  which is proper lower semicontinuous and bounded from below. Then for  $\varepsilon > 0$  and  $u \in X$ , there exists  $v \in X$  such that*

$$f(v) \leq f(u) - \varepsilon \int_{d(z_0, u)}^{d(z_0, u) + d(u, v)} \frac{dt}{1 + h(t)} - \varepsilon \int_{d(z_0, v)}^{d(z_0, v) + d(u, v)} \frac{dt}{1 + h(t)}$$

and

$$f(w) > f(v) - \varepsilon \int_{d(z_0, v)}^{d(z_0, v) + d(v, w)} \frac{dt}{1 + h(t)} - \varepsilon \int_{d(z_0, w)}^{d(z_0, w) + d(v, w)} \frac{dt}{1 + h(t)}$$

for all  $w \in X$  with  $w \neq v$ .

### 3. ZHONG'S RESULT

In this section, using Theorem 2, we can easily prove the following Zhong's result in [17]. Compare the proof with Zhong's. We use Theorem 2 only one time.

**Theorem 3** (Zhong [17]). *Let  $X$  be a Banach space, and  $h$  a nondecreasing function from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + h(t)))dt = \infty$ . Let  $f$  be a function from  $X$  into  $(-\infty, +\infty]$  which is proper lower semicontinuous. Assume that  $f$  is Gâteaux differentiable at every point  $x \in X$  with  $f(x) \in \mathbb{R}$ . If*

$$\alpha := \lim_{r \rightarrow \infty} \inf_{\|x\| \geq r} f(x) \in \mathbb{R},$$

*then there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_n \|x_n\| = \infty$ ,  $\lim_n f(x_n) = \alpha$ , and*

$$\lim_{n \rightarrow \infty} \|f'(x_n)\| (1 + h(\|x_n\|)) = 0.$$

*Remark.* In [17], the continuity of  $h$  is needed. We discuss this condition in Section 5.

In the proof of Theorem 3, we use the following lemma, which is well known.

**Lemma 1.** Suppose that  $c \geq 0$ ,  $\delta > 0$ ,  $v \in X$ ,  $f(v) \in \mathbb{R}$  and either of the following holds:

- $f(w) \geq f(v) - c \|v - w\|$  for all  $w \in X$  with  $0 < \|v - w\| < \delta$ ; or
- $f(w) \leq f(v) + c \|v - w\|$  for all  $w \in X$  with  $0 < \|v - w\| < \delta$ .

Then  $\|f'(v)\| \leq c$ .

*Proof of Theorem 3.* We shall only show the following: For every  $\varepsilon > 0$ , there exists  $v \in X$  satisfying  $\|v\| \geq 1/\varepsilon$ ,  $|f(v) - \alpha| \leq \varepsilon$ , and  $\|f'(v)\| (1 + h(\|v\|)) \leq \varepsilon$ . Fix  $\varepsilon > 0$ . Define a function  $\theta$  from  $[0, \infty)$  into itself by

$$(1) \quad \theta(t) = 1 + 2h(t+1)$$

for  $t \in [0, \infty)$ . Then it is obvious that  $\theta$  is nondecreasing, and we have

$$\int_0^\infty \frac{dt}{1 + \theta(t)} = \frac{1}{2} \int_0^\infty \frac{dt}{1 + h(t+1)} = \frac{1}{2} \int_1^\infty \frac{dt}{1 + h(t)} = \infty.$$

We also define a function  $g$  from  $X$  into  $(-\infty, +\infty]$  by

$$g(x) = \max \{f(x), \alpha - 2\varepsilon\}$$

for  $x \in X$ . Then it is obvious that  $g$  is proper lower semicontinuous and bounded from below. We next choose  $r, r' \in \mathbb{R}$  with  $1/\varepsilon < r < r'$ ,  $1 < r$ ,

$$\inf_{\|x\| \geq r} f(x) > \alpha - \varepsilon, \quad \text{and} \quad \int_r^{r'} \frac{dt}{1 + \theta(t)} = 3.$$

We also choose  $u \in X$  with  $\|u\| > r'$  and  $f(u) < \alpha + \varepsilon$ . We note that  $g(u) = f(u)$  because of  $\|u\| > r$ . Then by Theorem 2, there exists  $v \in X$  such that

$$(2) \quad g(v) \leq g(u) - \varepsilon \int_{\|u\|}^{\|u\| + \|u-v\|} \frac{dt}{1 + \theta(t)} - \varepsilon \int_{\|v\|}^{\|v\| + \|u-v\|} \frac{dt}{1 + \theta(t)}$$

and

$$(3) \quad g(w) > g(v) - \varepsilon \int_{\|v\|}^{\|v\| + \|v-w\|} \frac{dt}{1 + \theta(t)} - \varepsilon \int_{\|w\|}^{\|w\| + \|v-w\|} \frac{dt}{1 + \theta(t)}$$

for all  $w \in X$  with  $w \neq v$ . Arguing by contradiction, we assume that  $\|v\| < r$ . From (2), we have

$$\begin{aligned} \alpha - 2\varepsilon &\leq g(v) \leq g(u) - \varepsilon \int_{\|v\|}^{\|v\| + \|u-v\|} \frac{dt}{1 + \theta(t)} \\ &\leq g(u) - \varepsilon \int_{\|v\|}^{\|u\|} \frac{dt}{1 + \theta(t)} \leq g(u) - \varepsilon \int_r^{r'} \frac{dt}{1 + \theta(t)} \\ &= f(u) - 3\varepsilon < \alpha - 2\varepsilon. \end{aligned}$$

This is a contradiction. Therefore we obtain  $\|v\| \geq r > 1/\varepsilon$ . Thus we have  $g(v) = f(v)$  and

$$\alpha - \varepsilon < \inf_{\|x\| \geq r} f(x) \leq f(v) \leq f(u) < \alpha + \varepsilon.$$

This implies  $|f(v) - \alpha| \leq \varepsilon$ . From (3) and nondecreasingness of  $\theta$ , we have

$$g(w) > g(v) - \left( \frac{\varepsilon}{1 + \theta(\|v\|)} + \frac{\varepsilon}{1 + \theta(\|w\|)} \right) \|v - w\|$$

for  $w \in X$  with  $w \neq v$ . Since  $f$  is lower semicontinuous and  $f(v) > \alpha - 2\varepsilon$ , there exists  $\delta \in (0, 1)$  such that  $f(w) > \alpha - 2\varepsilon$  for  $w \in X$  with  $\|v - w\| < \delta$ . Hence, for  $w \in X$  with  $0 < \|v - w\| < \delta$ , since  $g(w) = f(w)$  and

$$\|w\| \geq \|v\| - \|v - w\| > \|v\| - \delta > \|v\| - 1 > 0,$$

we have

$$\begin{aligned} f(w) &> f(v) - \left( \frac{\varepsilon}{1 + \theta(\|v\|)} + \frac{\varepsilon}{1 + \theta(\|v\| - 1)} \right) \|v - w\| \\ &\geq f(v) - \frac{2\varepsilon}{1 + \theta(\|v\| - 1)} \|v - w\| \\ &= f(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|. \end{aligned}$$

So by Lemma 1, we have  $\|f'(v)\| (1 + h(\|v\|)) \leq \varepsilon$ . This completes the proof.  $\square$

As a direct consequence of Theorem 3, we obtain the following.

**Theorem 4** (Zhong [17]). *Let  $X$  be a Banach space. Let  $f$  be a function from  $X$  into  $(-\infty, +\infty]$  which is proper lower semicontinuous and bounded from below. Assume that  $f$  is Gâteaux differentiable at every point  $x \in X$  with  $f(x) \in \mathbb{R}$ , and  $f$  satisfies the weak Palais-Smale condition. Then  $f$  is coercive.*

*Remark.* We can weaken the condition that  $f$  satisfies the weak Palais-Smale condition as follows: Every sequence  $\{x_n\}$  in  $X$  such that  $\{f(x_n)\}$  is bounded and  $\lim_n \|f'(x_n)\| (1 + h(\|x_n\|)) = 0$  contains a bounded subsequence.

#### 4. COERCIVITY OF $|f|$

In this section, we discuss the coercivity of  $|f|$ .

The following is a generalization of the result in Caklovic, Li and Willem [3].

**Theorem 5.** *Let  $X$  be a Banach space, and  $h$  a nondecreasing function from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + h(t)))dt = \infty$ . Let  $f$  be a continuous function from  $X$  into  $\mathbb{R}$ . Assume that  $f$  is Gâteaux differentiable at every point  $x \in X$ . If there exists  $\gamma \in \mathbb{R}$  such that  $\{x \in X : f(x) = \gamma\}$  is bounded, and*

$$\alpha := \lim_{r \rightarrow \infty} \inf_{\|x\| \geq r} |f(x) - \gamma| \in \mathbb{R},$$

*then there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_n \|x_n\| = \infty$ ,  $\lim_n |f(x_n) - \gamma| = \alpha$ , and*

$$\lim_{n \rightarrow \infty} \|f'(x_n)\| (1 + h(\|x_n\|)) = 0.$$

*Proof.* We put  $g(x) = |f(x) - \gamma|$  for all  $x \in X$ . We shall only show the following: For every  $\varepsilon > 0$ , there exists  $v \in X$  satisfying  $\|v\| \geq 1/\varepsilon$ ,  $|g(v) - \alpha| \leq \varepsilon$ , and  $\|f'(v)\| (1 + h(\|v\|)) \leq \varepsilon$ . Fix  $\varepsilon > 0$ . Define a function  $\theta$  from  $[0, \infty)$  into itself by (1). We next choose  $r, r' \in \mathbb{R}$  with  $1/\varepsilon < r < r'$ ,  $1 < r$ ,  $g(x) > 0$  for  $x \in X$  with  $\|x\| \geq r$ ,

$$\inf_{\|x\| \geq r} g(x) > \alpha - \varepsilon, \quad \text{and} \quad \int_r^{r'} \frac{dt}{1 + \theta(t)} = \frac{\alpha + \varepsilon}{\varepsilon}.$$

We also choose  $u \in X$  with  $\|u\| > r'$  and  $g(u) < \alpha + \varepsilon$ . Then by Theorem 2, there exists  $v \in X$  with (2) and (3) for all  $w \in X$  with  $w \neq v$ . Arguing by contradiction, we assume that  $\|v\| < r$ . From (2), we have

$$0 \leq g(v) \leq g(u) - \varepsilon \int_r^{r'} \frac{dt}{1 + \theta(t)} = g(u) - (\alpha + \varepsilon) < 0.$$

This is a contradiction. Therefore we obtain  $\|v\| \geq r > 1/\varepsilon$  and hence  $g(v) > 0$ . We also have

$$\alpha - \varepsilon < \inf_{\|x\| \geq r} g(x) \leq g(v) \leq g(u) < \alpha + \varepsilon.$$

and hence  $|g(v) - \alpha| \leq \varepsilon$ . Since  $f$  is continuous and  $g(v) > 0$ , there exists  $\delta \in (0, 1)$  such that either of the following holds:

- $g(w) = +f(w) - \gamma$  for  $w \in X$  with  $\|v - w\| < \delta$ ; or
- $g(w) = -f(w) + \gamma$  for  $w \in X$  with  $\|v - w\| < \delta$ .

As in the proof of Theorem 3, we have

$$g(w) > g(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|$$

for  $w \in X$  with  $0 < \|v - w\| < \delta$ . In the former case, we obtain

$$f(w) > f(v) - \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.$$

In the latter case, we obtain

$$f(w) < f(v) + \frac{\varepsilon}{1 + h(\|v\|)} \|v - w\|.$$

So, by Lemma 1, we have  $\|f'(v)\| (1 + h(\|v\|)) \leq \varepsilon$  in both cases. This completes the proof.  $\square$

As a direct consequence of Theorem 5, we obtain the following.

**Theorem 6.** *Let  $X$  be a Banach space. Let  $f$  be a continuous function from  $X$  into  $\mathbb{R}$ . Assume that  $f$  is Gâteaux differentiable at every point  $x \in X$ , and  $f$  satisfies the weak Palais-Smale condition. If there exists  $\gamma \in \mathbb{R}$  such that  $\{x \in X : f(x) = \gamma\}$  is bounded, then  $|f|$  is coercive.*

*Remark.* We have the same remark of Theorem 4.

## 5. CONTINUITY OF $h$

In this section, we discuss the continuity of  $h$ .

Without the assumption of continuity of  $h$ , we can prove Theorem 3. However, Theorem 3 is not a generalization of Zhong's result because the following proposition holds. That is, Theorem 3 in this paper and Theorem 3.7 in [17] are equivalent. Also the two definitions of weak Palais-Smale condition in [17] and in this paper are equivalent.

**Proposition 2.** *Let  $h$  be a nondecreasing function from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + h(t))) dt = \infty$ . Then there exists a continuous nondecreasing function  $\theta$  from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + \theta(t))) dt = \infty$  and  $h(t) \leq \theta(t)$  for all  $t \in [0, \infty)$ .*

*Proof.* For  $t \in \mathbb{R}$ , we denote by  $[t]$  the maximum integer not exceeding  $t$ . Define a function  $\theta$  from  $[0, \infty)$  into itself by

$$\theta(t) = (1 - t + [t]) h([t] + 1) + (t - [t]) h([t] + 2)$$

for  $t \in [0, \infty)$ . Putting  $k = [t]$  and  $s = t - [t] \in [0, 1)$ , we have

$$\theta(k + s) = (1 - s) h(k + 1) + s h(k + 2).$$

It is obvious that  $\theta$  is continuous and nondecreasing. For  $t \in [0, \infty)$ , we have

$$\theta(t) \geq h([t] + 1) \geq h(t)$$

because  $t < [t] + 1$ . We also have

$$\int_0^\infty \frac{dt}{1 + \theta(t)} \geq \int_0^\infty \frac{dt}{1 + h([t] + 2)} \geq \int_0^\infty \frac{dt}{1 + h(t + 2)} = \int_2^\infty \frac{dt}{1 + h(t)} = \infty.$$

This completes the proof.  $\square$

Similarly, we can prove the following.

**Proposition 3.** *Let  $h$  be a nondecreasing function from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + h(t)))dt < \infty$ . Then there exists a continuous nondecreasing function  $\theta$  from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + \theta(t)))dt < \infty$  and  $\theta(t) \leq h(t)$  for all  $t \in [0, \infty)$ .*

*Proof.* Define a function  $\theta$  from  $[0, \infty)$  into itself by

$$\theta(t) = \begin{cases} h(0), & \text{if } t \leq 1, \\ (1 - t + [t]) h([t] - 1) + (t - [t]) h([t]), & \text{if } t \geq 1. \end{cases}$$

for  $t \in [0, \infty)$ . Then  $\theta$  is continuous, nondecreasing,  $\theta(t) \leq h([t]) \leq h(t)$  for  $t \in [0, \infty)$ , and  $h(t - 2) \leq h([t] - 1) \leq \theta(t)$  for  $t \in [2, \infty)$ . Hence

$$\int_2^\infty \frac{dt}{1 + \theta(t)} \leq \int_2^\infty \frac{dt}{1 + h(t - 2)} = \int_0^\infty \frac{dt}{1 + h(t)} < \infty.$$

This completes the proof.  $\square$

## 6. COUNTEREXAMPLES

In this section, we give examples, which say that we use conditions  $\int_0^\infty (1/(1 + h(t)))dt = \infty$  and the completeness of  $X$  in Theorem 3 and others.

**Example 2.** Put  $X := \mathbb{R}$  and let  $h$  be a nondecreasing function from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + h(t)))dt < \infty$ . Then there exists a differentiable function  $f$  from  $X$  into  $\mathbb{R}$  such that

$$\lim_{r \rightarrow \infty} \inf_{|x| \geq r} f(x) \in \mathbb{R} \quad \text{and} \quad |f'(x)| (1 + h(|x|)) \geq 1$$

for all  $x \in X$ .

*Proof.* By Proposition 3, there exists a continuous nondecreasing function  $\theta$  from  $[0, \infty)$  into itself such that  $\int_0^\infty (1/(1 + \theta(t)))dt < \infty$  and  $\theta(t) \leq h(t)$  for all  $t \in [0, \infty)$ . Define a function  $f$  from  $X$  into  $\mathbb{R}$  by

$$f(x) = \int_0^x \frac{-1}{1 + \theta(\max\{t, 0\})} dt$$



for  $x \in X$ . It is obvious that  $\lim_{x \rightarrow -\infty} f(x) = \infty$  and  $\lim_{x \rightarrow +\infty} f(x) \in \mathbb{R}$ . We also have

$$|f'(x)| (1 + h(|x|)) = \frac{1}{1 + \theta(\max\{x, 0\})} (1 + h(|x|)) \geq \frac{1 + h(|x|)}{1 + \theta(|x|)} \geq 1$$

for all  $x \in X$ . This completes the proof.  $\square$

**Example 3.** Let  $X$  be the normed linear space consisting of all functions  $x$  from  $\mathbb{N}$  into  $\mathbb{R}$  (i.e.,  $x$  is a real sequence) such that  $\{n \in \mathbb{N} : x(n) \neq 0\}$  is a finite subset of  $\mathbb{N}$ . Define a norm  $\|\cdot\|$  on  $X$  by  $\|x\| = \sum_{n=1}^{\infty} |x(n)|$  for all  $x \in X$ . Define a lower semicontinuous (not continuous), convex, and Gâteaux differentiable function  $f$  from  $X$  into  $\mathbb{R}$  by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \exp(2^n x(n))$$

for  $x \in X$ . Then

$$\lim_{r \rightarrow \infty} \inf_{\|x\| \geq r} f(x) = 0 \in \mathbb{R} \quad \text{and} \quad \|f'(x)\| \geq 1$$

for all  $x \in X$ .

*Proof.* It is obvious that  $f$  is convex and  $\lim_{r \rightarrow \infty} \inf\{f(x) : \|x\| \geq r\} = 0$ . By the definition of  $X$ ,  $f$  is Gâteaux differentiable and its derivative is given by

$$f'(x) = \sum_{n=1}^{\infty} \exp(2^n x(n)) e_n$$

for all  $x \in X$ , where  $\{e_n\}$  is the canonical basis of  $X$ . Thus, we have

$$\|f'(x)\| = \sup \{ \exp(2^n x(n)) : n \in \mathbb{N} \} \geq \exp(0) = 1$$

for all  $x \in X$ . Fix  $x \in X$  and define a sequence  $\{x_n\}$  in  $X$  by

$$x_n(k) = \begin{cases} x(k), & \text{if } k \neq n, \\ 1/n, & \text{if } k = n \end{cases}$$

for  $n \in \mathbb{N}$ . Since  $\|x - x_n\| = 1/n$  for large  $n \in \mathbb{N}$ ,  $\{x_n\}$  converges to  $x$ . Since

$$\frac{2^{n-1}}{n^2} \leq \frac{1}{2^n} \left( 1 + \frac{2^n}{n} + \left( \frac{2^n}{n} \right)^2 / 2 \right) \leq \frac{1}{2^n} \exp\left(\frac{2^n}{n}\right) \leq f(x_n)$$

for  $n \in \mathbb{N}$ , we have  $\lim_n f(x_n) = \infty$ . This implies  $f$  is not continuous everywhere. We finally show that  $f$  is lower semicontinuous. Let  $\{x_n\}$  be a sequence in  $X$  converging to some  $x \in X$ . We fix  $\varepsilon > 0$  and choose  $\nu \in \mathbb{N}$  such that  $2^{-\nu} < \varepsilon$  and  $x(n) = 0$  for every  $n \in \mathbb{N}$  with  $n \geq \nu$ . Define functions  $g$  and  $h$  from  $X$  into  $(0, \infty)$  by

$$g(y) = \sum_{n=1}^{\nu} \frac{1}{2^n} \exp(2^n y(n)) \quad \text{and} \quad h(y) = \sum_{n=\nu+1}^{\infty} \frac{1}{2^n} \exp(2^n y(n))$$

for  $y \in X$ . Then it is obvious that  $f = g + h$ ,  $g$  is continuous and  $h(x) = 2^{-\nu} < \varepsilon$ . We have

$$f(x) = g(x) + h(x) \leq g(x) + \varepsilon = \lim_{n \rightarrow \infty} g(x_n) + \varepsilon \leq \liminf_{n \rightarrow \infty} f(x_n) + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we have  $f(x) \leq \liminf_n f(x_n)$ . Therefore  $f$  is lower semicontinuous. This completes the proof.  $\square$

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